COUNTABLY GENERATED FAMILIES

R. DANIEL MAULDIN

ABSTRACT. This paper discusses some interrelationships between various statements involving sets generated by rectangles, universal spaces, and real-valued measures on the continuum. Borel sets on ordinal spaces are also discussed.

Definition. Let \( E \) denote a set. If \( A \) is a subset of \( E^2 = E \times E \), let \( A_x = \{ y | (x, y) \in A \} \), for each \( x \) in \( E \) and let \( A^y = \{ x | (x, y) \in A \} \), for each \( y \) in \( E \). Let \( R \) be the family of all sets of the form \( A \times B \) in \( E^2 \).

If \( G \) is a subset of \( 2^E \), let \( G_0 \) be \( G \) and for each ordinal \( \alpha \), let \( G_\alpha \) be the family of all countable unions (intersections) of sets in \( \bigcup_{\gamma < \alpha} G_\gamma \) if \( \alpha \) is odd (even). Limit ordinals will be considered even. Of course, \( G_{\omega_1} \) is the smallest family including \( G \) which is closed under countable unions and intersections; \( G_{\omega_1} \) is the Borel lattice generated by \( G \). It can be checked that if for each \( A \in G \), \( A' \in G_{\omega_1} \), then \( G_{\omega_1} \) is closed under complements and \( G_{\omega_1} \) is then the \( \sigma \)-algebra or Borel algebra generated by \( G \).

In [1], a study is made of the Borel lattice (algebra) generated by the family \( R \) and a number of the results stated in that paper will be used here. In particular, if \( |E| > \aleph \), then the main diagonal in \( E \times E \) is not in \( R_{\omega_1} \). If \( |E| < \omega_1 \), then \( 2^{E^2} = R_{\omega_1} = R_{\omega_0} \). (It should be noted that throughout this paper, the Axiom of Choice is assumed and cardinals are regarded as initial ordinals.)

Kunen [4] investigated the family \( R_{\omega_1} \) when \( |E| \leq \aleph_0 \). He showed that Martin's Axiom implies \( R_{\omega_0} = 2^{E^2} \), if \( |E| \leq \aleph_0 \).

Recently, Franklin Tall and Kenneth Kunen have constructed a model of ZFC in which Martin's Axiom fails and yet \( 2^{E^2} = R_{\omega_1} \), if \( |E| \leq \aleph_0 \). R. Mansfield [18] and B. V. Rao [19], [20] have also studied the sets generated by rectangles and have solved some of Ulam’s problems with their aid.

There are a number of interesting consequences which follow from the assumption that \( 2^{E^2} = R_{\omega_1} \) and from the techniques which have been used in the study of families generated by \( R \).

Results.

THEOREM 1. Suppose \( 2^{E^2} = R_{\omega_0} \), where \( |E| = \aleph_0 \), and if \( \lambda < \aleph_0 \), then \( 2^\lambda \leq \aleph_0 \). Then there is a subset \( M \) of \([0,1] \) such that the Banach space \( B_1(M) \) consisting of the bounded real-valued functions of Baire's class 1 defined on \( M \) and under the uniform norm is universal for all Banach spaces of cardinality \( \aleph_0 \).

Received by the editors November 15, 1974.

AMS (MOS) subject classifications (1970). Primary 04A15, 04A30; Secondary 28A05.

Key words and phrases. Rectangles, continuum hypothesis, Martin's Axiom, universal space, measurable cardinal, ray of ordinals, Borel sets.
PROOF. It is known that the condition $\lambda < c$ implies $2^\lambda \leq c$ is equivalent to $c = c^{\mathbb{C}}$. It is known that $c = c^{\mathbb{C}}$ implies there is a zero-dimensional compact $T_2$ space $U$ of weight $c$ such that if $Y$ is a compact $T_2$ space weight $\leq c$, then $Y$ is a continuous image of $U$ [16], [17, p. 131]. If $X$ is a Banach space and $|X| = c$, then $X$ can be embedded in a space $C(S)$ where $S$ is a compact $T_2$ space and $|C(S)| = c$. But if $|C(S)| \leq c$, then weight of $S$ is $\leq c$. Thus, $X$ can be embedded in $C(U)$. Since $U$ has weight $c$, $|C(U)| \leq c$. Now by Theorem 4.4 of [7], there is a subset $M$ of $[0,1]$ such that $B_1(M)$ is universal for all Banach spaces of cardinality $c$.

Question. Does the existence of a universal Banach space of cardinality $c$ imply $c = c^{\mathbb{C}}$ or that there is a compact $T_2$ space, $U$, of weight $c$ so that every compact $T_2$ space of weight $c$ is a continuous image of $U$?

There are a number of other consequences of the assumption that $R_2 = R_{\omega_1} = 2^{E^2}$ by itself or together with a cardinality condition. For example, if $2^\kappa = 2^{\omega_1}$ and $R_2 = 2^{E^2}$, then there exists a $Q$-set [7, Theorem 4.5].

In [4], Kunen showed that $R_{\omega_1} = 2^{E^2}$ implied that $|E|$ is not a real-valued measurable cardinal. As Kunen points out, his argument is a variant of the known fact that a well-ordering of the real numbers is not Lebesgue measurable [11]. We shall generalize this argument as follows:

**Theorem 2.** Let $|E| = c$. For each positive integer $i$, statement $i$ implies statement $i + (1)$:

1. $2^{\omega_0} = \mathbb{N}_1$;
2. Martin's Axiom;
3. $2^{E^2} = R_\omega = R_{\omega_1}$;
4. $2^{E^2} = R_{\omega_1}$;
5. there is a countable ordinal $\alpha$ such that if $H$ is a family of $c$ subsets of $E$, then there is a countable family $G$ of subsets of $E$ such that $H \subseteq G_\alpha$;
6. there is a countable ordinal $\alpha$ such that if $H$ is a family of $c$ subsets of $E$ and each member of $H$ has cardinality $< c$, then there is a countable family $G$ such that $H \subseteq G_\alpha$;
7. if $H$ is a family of $c$ subsets of $E$ and each member of $H$ has cardinality $< c$, then there is a countable family $G$ such that $H$ is a subcollection of the Borel algebra generated by $G$;
8. if $W$ is a subset of $E^2$, then $W$ is the union of a subfamily $G$ of $R_{\omega_1}$ such that the cardinality of $G$ is not real-valued measurable;
9. if $\kappa$ is a real-valued measurable cardinal, then $\kappa > c$.

**Proof.** For the first three implications, see Kunen [4]. As has been mentioned, 3 $\Rightarrow$ 2. Clearly 3 $\Rightarrow$ 4, and in [1] it is shown that 4 $\Rightarrow$ 5. The question "Does 4 $\Rightarrow$ 3?" was raised in [1]. Clearly 5 $\Rightarrow$ 6.

We shall now show that 6 $\Rightarrow$ 4.

Let $E$ be well ordered into an initial type; $E = \{x_1, x_2, \ldots, x_\alpha, \ldots | \alpha < c \}$. Let $A = \{(x_\alpha, x_\gamma) | \alpha < \gamma \}$ and let $B = E^2 - A$. For each $y$, $|A^y| < c$ and for each $x$, $|B_x| < c$.

Let $Z \subseteq E^2$. Then $Z = (Z \cap A) \cup (Z \cap B)$. Let $H = \{(Z \cap A)^y | y \in E \}$. Each member of $H$ has cardinality $< c$. Thus, if 6 holds, there is a countable family of $G$ of subsets of $E$ and a countable ordinal $\alpha$ such that $H \subseteq G_\alpha$. It follows directly from Theorem 3 of [1] that $Z \cap A \in R_{\omega_1}$.

Similarly, $Z \cap B \in R_{\omega_1}$, So, 6 $\Rightarrow$ 4.
It should be noted that this argument may be used to show that if $|T| = \omega_1$, then every subset of $T^2$ is in the family $R_{\omega_0}$. This fact is proven by different means by Kunen [4].

Also, it should be pointed out that if statement 5 holds, then if $H$ is a family of $c$ subsets of $E$, there is a countable family $G$ of subsets of $E$ such that $H \subseteq \bigcup_{\alpha < \omega} G_{\alpha}$. This problem was raised by Ulam and Rothberger [9], [14].

Clearly, $6 \rightarrow 7$. It is unknown to the author whether $7 \rightarrow 6$.

We shall now show that $7 \rightarrow 8$.

Let $A$ and $B$ be the sets described above. Let $W \subseteq E^2$ and let $H = \{(A \cap W)^y \mid y \in E\}$. Clearly, each member of $H$ has cardinality $< c$ and $|H| \leq c$. Thus, from 7, there is a $G$, $|G| \leq \omega_0$ such that $H \subseteq \bigcup_{\gamma < \omega} G_{\gamma}$.

For each $\gamma < \omega_1$, let $K_{\gamma} = \{(x, y) \mid (x, y) \in A \cap W \text{ and } (A \cap W)^y \in G_{\gamma}\}$. It follows from Theorem 3 of [1] that $K_{\gamma} \in R_{\gamma}$.

Thus, $W \cap A = \bigcup_{\gamma < \omega} K_{\gamma}$. There is a similar argument for $W \cap B$ and certainly $7 \rightarrow 8$ since $\omega_1$ is not a real-valued measurable cardinal.

Finally, we show $8 \rightarrow 9$. We argue indirectly. Suppose $\kappa, \kappa \leq c$, is real-valued measurable and 8 holds. Let $\omega_\alpha$ be the first real-valued measurable cardinal and let $S = \{\gamma \mid \gamma < \omega_\alpha\}$ and let $A = \{(x, y) \mid (x, y) \in S \times S \text{ and } x \text{ precedes } y\}$.

It follows from statement 8 that there is a subfamily $G = \{A_{\gamma}\}_{\gamma < \lambda}$ of the Borel field generated by the rectangles over $S$ such that $A = \bigcup_{\gamma < \lambda} A_{\gamma}$ and $\lambda$ is not real-valued measurable.

Let $\mu$ be a free probability measure on $\omega_\alpha$ which is $\omega_\alpha$-additive. For each $\gamma < \lambda$, $A_{\gamma}$ is $\mu \times \mu$-measurable. We calculate the measure of $A_{\gamma}$ by Fubini's theorem.

$$\mu \times \mu(A_{\gamma}) = \int_{S \times S} \chi_{A_{\gamma}} d(\mu \times \mu) = \int_S \left( \int_S \xi_{A_{\gamma}}(x, y) d\mu(x) \right) d\mu(y).$$

But, for each $y$, $\int_S \xi_{A_{\gamma}}(x, y) d\mu(x) = \mu(A_{\gamma}) = 0$. Thus, $(\mu \times \mu)(A_{\gamma}) = 0$.

For each $\gamma < \lambda$, let $P_{\gamma} = \{x \mid \mu((A_{\gamma})_x) > 0\}$. It follows from Fubini's theorem that each $P_{\gamma}$ has $\mu$-measure 0.

However, for each $x \in S$, $\mu(A_{x}) > 0$ and $A_{x} = \bigcup_{\gamma < \lambda} (A_{\gamma})_x$. Thus, $\bigcup_{\gamma < \lambda} P_{\gamma} = S$. But, since $\mu$ is $\omega_\alpha$-additive, $\mu(S) = 0$. This contradiction completes the proof of the theorem.

Remark 1. The theorem that $8 \rightarrow 9$ was also proven by E. Fisher in his thesis [2]. In fact, Fisher showed that no well-ordering of $\omega_\alpha$ is in the $\omega_\alpha$-algebra generated by $R$. The author was unaware of this and thanks the referee for pointing this out and for making a number of other helpful comments.

Remark 2. In the first issue of Colloquium Mathematicum, Banach showed that the continuum hypothesis implies that there is a countable family of subsets of $I$, the unit interval, such that Lebesgue measure cannot be extended from the Lebesgue measurable sets to a $\sigma$-algebra containing these sets. The same result holds under Martin's Axiom. Any countable family $(E_n)_{n=1}^\infty$ such that a well-ordering of $I$ (regarded as a subset of $I \times I$) is in the $\sigma$-algebra generated by the rectangles $A_n \times A_m$ will suffice. The argument is the same as above, in view of the fact that Lebesgue measure is $c$-additive under Martin's Axiom [6].
As mentioned earlier, it is apparently unknown whether \(R_{\omega_1} = 2^{E^2}\) implies \(R_\beta = R_{\omega_1} = 2^{E^2}\). In fact, it is apparently unknown whether there is any family of sets \(G\) such that \(G_\alpha = G_{\alpha + 1}\), but \(G_\beta \neq G_\gamma\) for \(\beta < \alpha\) and \(\alpha > 3\) [3]. It is known that the Baire order of compact \(T_2\) spaces is either 0, 1 or \(\omega_1\) (here \(G\) is the family of all closed \(O_\delta\) sets) [8]. It is apparently unknown what the Borel order of a compact \(T_2\) space may be (here \(G\) is the family of sets which are the intersection of an open set and a closed set).

We now describe the Borel subsets of the ordinal spaces \([0, \alpha]\) provided with the order topology. First, in Theorem 4, the Borel subsets of \([0, \omega_1]\) are described. This theorem was proven by M. Bhaskara Rao and K. P. S. Bhaskara Rao [21].

**Theorem 3.** Every Borel subset of \([0, \omega_1]\) can be expressed as the union of countable many sets, each of which is the intersection of an open set and a closed set.

**Proof.** Let \(\mathcal{M}\) be the \(\sigma\)-algebra of all subsets \(E\) of \([0, \omega_1]\) such that \(E\) or \(E^c\) contains a closed unbounded subset of \([0, \omega_1]\). Clearly \(\mathcal{M}\) contains the open sets and the closed sets.

Suppose \(E \in \mathcal{M}\) and \(E^c\) contains a closed unbounded set \(F_0\). Let \(\{V_\alpha\}_{\alpha \in A}\) be the set of all order components of the complement of \(F\). Then \(E \cap V_\alpha\) is countable: \(E \cap V_\alpha = \{x_{\alpha n}\}_{n=1}^{\infty}\). Let \(K_\alpha = \{x_{\alpha n}\}_{\alpha \in A}\). For each \(\alpha\), \(K_\alpha\) is closed in \(F^c\). Thus, \(K_\alpha = F_\alpha \cap V\), where \(F_\alpha = K_\alpha\) and \(V = F^c\) and \(E = \bigcup_{n=1}^{\infty} K_\alpha\).

If \(E \in \mathcal{M}\) and \(E\) contains a closed unbounded set \(F_0\), then as before \(E - F_0 = \bigcup_{n=1}^{\infty} (F_n \cap V)\) and \(E = F_0 \cup \bigcup_{n=1}^{\infty} (F_n \cap V)\) where \(V = F^c_0\).

Thus, \(\mathcal{M}\) is the family of all Borel subsets of \([0, \omega_1]\) and \(E\) is a Borel subset of \([0, \omega_1]\) if and only if \(E\) or \(E^c\) contains a closed unbounded set.

**Remark 3.** In contrast with the classical development, the smallest family containing the closed subsets of \([0, \omega_1]\) which is closed under countable unions and intersections is not the Borel algebra generated by the closed sets. In fact, let \(H = \{x | x\) is countable or \(x\) contains an unbounded closed set \}. Then \(H\) contains all the closed sets, \(H_\sigma = H_\delta = H\), and yet \(\mathcal{M} \neq \mathcal{M}\).

**Remark 4.** It is known that the \(\sigma\)-algebra generated by Borel measurable rectangles in \([0, \omega_1] \times [0, \omega_1]\) does not include all Borel subsets of \([0, \omega_1] \times [0, \omega_1]\). In fact, the sets \(D_1 = \{(x, y) | y > x\}\) and \(D_2 = \{(x, y) | y < x\}\) are disjoint open sets which are not measurable with respect to the outer measure induced by the gauge \(g(A \times B) = \mu(A) \cdot \mu(B)\), where \(\mu\) is Dieudonné's measure: \(\mu(E) = 1\), if \(E\) contains a closed unbounded set and \(\mu(E) = 0\), otherwise.

We have

\[
\mu^*_g(E) = \inf \left\{ \sum_{n=1}^{\infty} g(A_n \times B_n) | \bigcup_{n=1}^{\infty} A_n \times B_n \supset E \right\}.
\]

It follows that \(\mu^*_g = 0, 1\)-valued. We show \(\mu^*_g(D_1) = \mu^*_g(D_2) = 1\) to show that there are nonmeasurable Borel sets. If \(\mu^*_g(D_1) = 0\), then there is a sequence \(\{A_n \times B_n\}_{n=1}^{\infty}\) such that \(D_1 \subset \bigcup_{n=1}^{\infty} (A_n \times B_n)\) and for each \(n\), \(A_n\) or \(B_n\) fails to contain a closed unbounded set.
Let $A_{n_1}, A_{n_2}, \ldots$ be the sequence of all the $A_n$'s of $\mu$-measure 0. Let $F$ be a closed unbounded subset of $\cap_{i=1}^\infty A_{n_i}$. Let $x \in F$ and let $K = \{y : y > x\}$. Then $\{x\} \times K \subset D_1$ and $\{x\} \times K \subset \bigcup_{i=1}^\infty (A_{m_i} \times B_{m_i})$ where $A_{m_i}$'s contain unbounded closed sets. Then no $B_{m_i}$ contains an unbounded set and yet $K \subset \bigcup_{i=1}^\infty B_{m_i}$. This is a contradiction.

**Note.** The referee points out that $D_1$ and $D_2$ are not measurable in $\mu \times \mu$ follows immediately by Fubini's theorem, as in the proof of $8 \rightarrow 9$ in Theorem 2.

**Theorem 4.** Let $\alpha$ be an ordinal. Every Borel subset of $[0, \alpha)$ can be expressed as the union of countably many sets, each of which is the intersection of an open set and a closed set.

**Proof.** Clearly, the theorem holds for all ordinals $\alpha$, $\alpha \leq \omega_1$. It is also easy to show that if the theorem holds for the ordinal $\alpha$, then it holds for $\alpha + 1$.

So assume $\alpha$ is a limit ordinal and the theorem holds for all $\beta < \alpha$. We consider two cases.

**Case 1.** $\text{cf}(\alpha) = \alpha$.

In this case, let $\mathcal{M} = \{E : E \text{ or } E' \text{ contains a closed unbounded set and } \forall \gamma < \alpha, E \cap [0, \gamma) \text{ is Borel in } [0, \gamma)\}$. $\mathcal{M}$ is a $\sigma$-algebra and it contains both the closed and the open subsets of $[0, \alpha)$.

Suppose $E \in \mathcal{M}$ and $E' \supset F_0$, $F_0$ a closed unbounded set. Let $(I_\gamma)_{\gamma \in \Gamma}$ be the set of order components of $[0, \alpha) \setminus F_0$. Then $E \cap I_\gamma$ is Borel in $I_\gamma$. Thus, $E \cap I_\gamma = \bigcup_{n=1}^\infty (F_{n\alpha} \cap O_{n\alpha})$, where $F_{n\alpha}$ is closed in $I_\gamma$ and $O_{n\alpha}$ is open in $I_\gamma$.

For each $n$, let $F_n = \bigcup_{\gamma \in \Gamma} F_{n\gamma}$ and let $U_n = \bigcup_{\gamma \in \Gamma} O_{n\gamma}$.

It follows that $E = \bigcup_{\gamma \in \Gamma} (E \cap I_\gamma) = \bigcup_{n=1}^\infty (F_n \cap U_n)$.

If $E \in \mathcal{M}$ and $E \supset F_0$, $F_0$ a closed unbounded set, then $(E \setminus F_0) \supset F_0$ and we obtain $E = F_0 \cup \bigcup_{n=1}^\infty (F_n \cap U_n)$.

**Case 2.** $\text{cf}(\alpha) = \tau < \alpha$.

In this case fix a set $F_0 = \{\gamma_n\}_{\beta < \tau}$ running through $\alpha$ and such that $F_0$ is closed. Let $(I_\sigma)_{\sigma \in \Sigma}$ be the set of order components of $F_0'$. If $E$ is Borel in $[0, \alpha)$, then $E \cap F_0$ is Borel in $F_0$ and $E \cap F_0$ is Borel in $I_\sigma$. Since $\tau < \alpha$, $E \cap F_0 = \bigcup_{n=1}^\infty (\mathcal{K}_n \cap V_n)$, where $\mathcal{K}_n$ is closed in $F_0$ and $V_n$ is open in $F_0$. Thus, $E \cap F_0 = \bigcup_{n=1}^\infty F_{2n} \cap U_{2n}, F_2 \cap U_{2n}$ closed in $[0, \alpha)$ and $U_{2n}$ open in $[0, \alpha)$. For each $\sigma \in \Sigma$, $E \cap I_\sigma = \bigcup_{n=1}^\infty (F_{n\alpha} \cap U_{n\alpha})$. Let $F_{2n-1} = \bigcup_{\sigma \in \Sigma} F_{n\sigma}$ and $U_{2n-1} = \bigcup_{\sigma \in \Sigma} U_{n\sigma}$. It follows that $E \cap F_0 = \bigcup (F_{2n-1} \cap U_{2n-1})$ and $E = \bigcup_{n=1}^\infty (F_n \cap U_n)$. Q.E.D.

Thus, if one considers the compact $T_2$ space $[0, \alpha]$, it has Borel order 1 no matter what ordinal $\alpha$ is.

**Problem.** Does this result hold for all compact scattered $T_2$ spaces? Is the Borel order of the other compact $T_2$ spaces $\omega_1$?

In [21], it is shown that there is no nonatomic, countably additive, finite measure defined on the Borel subsets of $[0, \omega_1)$. We generalize this in the next theorem.

**Theorem 5.** If there is no real-valued measurable cardinal $\kappa$ with $\kappa \leq \alpha$, then every countably additive finite measure defined on the Borel subsets of $[0, \alpha)$ is purely atomic.
PROOF. Let us assume the contrary. Let us assume that $\alpha$ is the first ordinal for which such a measure exists and that $\mu$ is a nonatomic probability measure defined on the Borel subsets of $[0, \alpha)$.

Notice that if $E$ is a Borel subset of $[0, \alpha)$ such that $E$ is Borel isomorphic to some space $[0, \beta)$, with $\beta < \alpha$, then $\mu(E) = 0$. Next notice that if $F$ is a closed cofinal subset of $[0, \alpha)$, then the open set $U = F'$ has measure zero. This can be seen as follows: First let $\varphi$ be a 1-1 map from $[0, \beta)$, for some $\beta \leq \alpha$, onto the set of order components of $U$. Define $\nu$ on each subset $W$ of $[0, \beta)$ by $
(W) = \mu(\bigcup \{\varphi(\gamma) : \gamma \in W\})$. Then $\nu$ is a free countably additive finite measure defined on all subsets of $[0, \beta)$. Therefore $\nu([0, \beta]) = \mu(U) = 0$.

Suppose $E$ is a Borel set which fails to contain a closed cofinal set. By the previous theorem, $E = \bigcup_{i=1}^{\infty} (F_i \cap U_i)$, where for each $i$, $F_i$ is closed and $U_i$ is open. For each $i$, either $F_i$ or $U_i$ fails to contain a closed cofinal set. If $U_i$ does not contain such a set, then $\mu(U_i) = 0$. If $F_i$ fails to contain such a set, then $F_i$ is a subset of an open set not containing a closed cofinal set. Therefore, $\mu(E) = 0$.

Finally, notice that if $B$ is a Borel set, then either $B$ or $B'$ contains a closed cofinal subset. But, this implies that $\mu$ is purely atomic.

REMARK. It is known that every regular Borel measure on any ordinal space (or, more generally, any compact dispersed space) is concentrated on a countable set.

REFERENCES

11. W. Sierpinski, Sur les rapports entre l existence des intégrales $\int_0^1 f(x)dx$ et $\int_0^1 f(x)dy$ et $\int_0^1 dx$ $\int_0^1 f(x,y)dy$, Fund. Math. 1 (1920), 142–147.


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611